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Quasi-periodic and periodic solutions for dynamical systems related to Korteweg-de Vries equation

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Abstract. We consider quasi-periodic and periodic (cnoidal) wave solutions of a set of n-component dynamical systems related to Korteweg-de Vries equation. Quasi-periodic wave solutions for these systems are expressed in terms of Novikov polynomials. Periodic solutions in terms of Hermite polynomials and generalized Hermite polynomials for dynamical systems related to Korteweg-de Vries equation are found.

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1 Introduction

We consider the reduced Gaudin magnet [1,2]

$$L(\lambda)=\sum_{j=1}^{n}\frac{\mathcal{L}_j}{\lambda-a_j}+B(\lambda),\quad \mathcal{L}_j(\lambda)=\begin{pmatrix} S_j^3 & S_j^+ \\ S_j^- & -S_j^3\end{pmatrix},\ (1)$$

where S_j satisfy n independent copies of the standard sl(2) algebra

$$\{S_j^3,S_k^{\pm}\}=\pm 2\delta_{jk}S_k^{\pm},\qquad \{S_j^+,S_k^-\}=4\delta_{jk}S_k^3. \qquad (2)$$

Many dynamical systems, for example Garnier system, Neumann system and multidimensional Hénon-Heiles system [2] are interpreted as reduced Gaudin magnets [1,2]. New examples of reduced Gaudin magnets, generalized Garnier system, Rosochatius system and multidimensional generalized Hénon-Heiles system are discussed in [3]. All these systems are related to stationary hierarchy of Korteweg-de Vries (KdV) system of equations (see for example [4–9]). The aim of present paper is to present quasiperiodic and periodic solutions for dynamical systems related to KdV equation and for associated reduced Gaudin magnets. The authors have already discussed quasiperiodic and periodic solutions associated with Lamé and Treibich-Verdier potentials for the Garnier type system [10–13]. We also mention the method of constructing elliptic finite-gap solutions of the stationary KdV and AKNS hierarchy, based on a theorem due to Picard, proposed in [14–17], as well the method developed by Smirnov in a series of publications: the review paper [19] and [20]. Elliptic AKNS solutions have been characterized in [17] and Trebich-Verdier potentials were fully analyzed in [18].

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2 Novikov, Hermite and generalized Hermite polynomials

In present paper our construction is based on *Novikov polynomials* [21] $F(x,\lambda)$ in λ of degree n, which are solutions of the following nonlinear differential equation

$$\frac{1}{2}FF_{xx}-\frac{1}{4}F_x^2-(\lambda+u(x))F^2+\frac{1}{4}\nu^2(\lambda)=0, \qquad (3)$$

where $u(x)$ is a real finite-gap potential given by Its-Matveev formula [22] and $\nu(\lambda)$ is a polynomial of degree $2n+1$ whose zeros are the branch points of the curve $\nu^2=4\prod_{j=0}^{2n}(\lambda-\lambda_j)$ or in another form equation (3) is written by

$$\frac{1}{2}FF_{xx}-\frac{1}{4}F_x^2-(\lambda+u(x))F^2+\sum_{j=0}^{2n+1}\lambda^{2n+1-j}\tilde{c}_j=0, \qquad (4)$$

with $\tilde{c}_0=1$. We seek solution of equation (4) as follows [23], $F_0=c_0=1$

$$F=\sum_{k=0}^{n}\sum_{m=0}^{n-k}\lambda^k c_m F_{n-k-m},\ F_1=-\frac{1}{2}u(x), \qquad (5)$$

where for $l > 1$ we have

$$F_l = \frac{1}{8}\sum_{s=1}^{l-1}(2F_{xx;s}F_{l-s-1} - F_{x;s}F_{x;l-s-1} \\ -4F_sF_{l-s} - 4u(x)F_sF_{l-s-1}) - \frac{1}{2}u(x)F_{l-1}. \quad (6)$$

As a result for first $F_l, l = 1, \ldots, 4$ we have

$$F_1 = -\frac{1}{2}u, \qquad F_2 = -\frac{1}{8}u_{xx} + \frac{3}{8}u^2,$$
$$F_3 = \frac{5}{16}u_{xx}u + \frac{5}{32}u_x^2 - \frac{5}{16}u^3 - \frac{1}{32}u_{xxxx},$$
$$F_4 = \frac{21}{128}u_{xx}^2 - \frac{35}{64}u_{xx}u^2 + \frac{7}{32}u_xu_{xxx} - \frac{35}{64}uu_x^2 + \frac{35}{128}u^4 + \frac{7}{64}uu_{xxxx} - \frac{1}{128}u_{xxxxxx},$$

where the following relations hold on

$$c_1 = \frac{1}{2}\tilde{c}_1, \qquad c_2 = \frac{1}{2}\tilde{c}_2 - \frac{1}{8}\tilde{c}_1^2,$$
$$c_3 = -\frac{1}{4}\tilde{c}_2\tilde{c}_1 + \frac{1}{16}\tilde{c}_1^3 + \frac{1}{2}\tilde{c}_3,$$
$$c_4 = \frac{1}{2}\tilde{c}_4 + \frac{3}{16}\tilde{c}_2\tilde{c}_1^2 - \frac{1}{4}\tilde{c}_3\tilde{c}_1 - \frac{1}{8}\tilde{c}_2^2 - \frac{5}{128}\tilde{c}_1^4.$$

The first few Novikov polynomials explicitly read,

$$F = \lambda - \frac{1}{2}u + \frac{1}{2}\tilde{c}_1, \qquad n = 1$$
$$F = \lambda^2 + (-\frac{1}{2}u + \frac{1}{2}\tilde{c}_1)\lambda - \frac{1}{8}u_{xx} + \frac{3}{8}u^2 - \frac{1}{4}\tilde{c}_1u + \frac{1}{2}\tilde{c}_2 - \frac{1}{8}\tilde{c}_1^2, \qquad n = 2$$
$$F = \lambda^3 + (-\frac{1}{2}u + \frac{1}{2}\tilde{c}_1)\lambda^2 + (-\frac{1}{8}u_{xx} + \frac{3}{8}u^2 - \frac{1}{4}\tilde{c}_1u + \frac{1}{2}\tilde{c}_2 - \frac{1}{8}\tilde{c}_1^2)\lambda + \frac{5}{16}uu_{xx} + \frac{5}{32}u_x^2 - \frac{5}{16}u^3 - \frac{1}{32}u_{xxxx} - \frac{1}{16}\tilde{c}_1u_{xx} + \frac{3}{16}\tilde{c}_1u^2 - \frac{1}{4}u\tilde{c}_2 + \frac{1}{16}u\tilde{c}_1^2 - \frac{1}{4}\tilde{c}_2\tilde{c}_1 + \frac{1}{16}\tilde{c}_1^3 + \frac{1}{2}\tilde{c}_3, \qquad n = 3\cdot$$

When $u(x)$ is n-gap Lamé potential $n(n+1)\wp(x)$ the Novikov polynomials are reduced to *Hermite polynomials.* When $u(x)$ are Treibich-Verdier potentials we have obtained a new polynomials called *generalized Hermite polynomials. Lamé polynomials* can be derived from Hermite polynomials when $\lambda = \lambda_j$, λ_j being the branch points. A special case of Lamé polynomials are known as *associated Legendre polynomials.* All the above mentioned polynomials are used to derive exact solutions of dynamical systems related to KdV equation.

Assuming that Novikov polynomial depend on "additional parameter" time t, the zero curvature representation for KdV hierarchy of equations have the following form

$$M_t(\lambda) - L_x(\lambda) + [M(\lambda), L(\lambda)] = 0, \quad (7)$$

where matrices L and M are given by

$$M(\lambda) = \begin{pmatrix} 0 & 1 \\ Q(x,t,\lambda) & 0 \end{pmatrix} \qquad L(\lambda) = \begin{pmatrix} -\frac{1}{2}F_x(x,t,\lambda) & F(x,t,\lambda) \\ -\frac{1}{2}F_{xx}(x,t,\lambda) + Q(x,t,\lambda)F(x,t,\lambda) & \frac{1}{2}F_x(x,t,\lambda) \end{pmatrix}.$$

The equation (7) is equivalent to

$$\frac{\partial Q}{\partial t} = -2\left[\frac{1}{4}\partial_x^3 - Q(x,t,\lambda)\partial_x - \frac{1}{2}Q_x(x,t,\lambda)\right]F(x,\lambda) \quad (8)$$

where $Q(x,t,\lambda) = u(x,t) + \lambda$ in the case of KdV hierarchy. Equation (8) is called the generating equation. The first few equations of the KdV hierarchy explicitly read,

$$u_t = u_x, \quad u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x + \frac{1}{2}\tilde{c}_1u_x,$$
$$u_t = \frac{1}{16}u_{xxxxx} - \frac{5}{8}uu_{xxx} - \frac{5}{4}uu_{xx} + \frac{15}{8}u^2u_x + \tilde{c}_1(\frac{1}{8}u_{xxx} - \frac{3}{4}uu_x) + \frac{1}{2}\tilde{c}_2u_x - \frac{1}{8}\tilde{c}_1^2u_x,$$
etc..

The Lax representation $L_x = [M, L]$ yields the hyperelliptic curve obtained by direct computation

$$\det(L(\lambda) - \mu \mathrm{I}_2) = 0, \quad (9)$$
$$\mu^2 = -\frac{1}{2}FF_{xx} + \frac{1}{4}F_x^2 + (\lambda + u)F^2 = -\frac{1}{4}\nu^2,$$

generating the Novikov polynomials related to stationary KdV hierarchy of equations.

3 Lax representation, integrals of motion and interpretation as reduced Gaudin magnets

3.1 Generalized Garnier type system

We consider the system [6,8,7,9]

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}q_j + \left(\sum_{k=1}^{n} q_k^2 - a_j\right)q_j - \frac{C_j^2}{q_j^3} = 0,\ j = 1, \ldots n, \quad (10)$$

where C_j, $j = 1, \dots n$ are free constants and a_j given points. The system (10) is a completely integrable Hamiltonian system related to the Korteweg-de Vries (KdV) equation with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{4}\left(\sum_{i=1}^{n} q_i^2\right)^2 - \frac{1}{2}\sum_{i=1}^{n} a_i q_i^2 + \frac{1}{2}\sum_{i=1}^{n} \frac{C_i^2}{q_i^2}, \tag{11}$$

where the variables $(q_i, p_i), i = 1, \dots n$, $p_i(x) = \mathrm{d}q_i(x)/\mathrm{d}x$, are the canonically conjugated variables with respect to the standard Poisson bracket, $\{\cdot\,;\,\cdot\}$.

This system has the Lax representation [9]

$$\frac{\mathrm{d}L(\lambda)}{\mathrm{d}x} = [M(\lambda), L(\lambda)],$$
$$L(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ Q(\lambda) & 0 \end{pmatrix}, \tag{12}$$

which is equivalent to (10), where $U(\lambda)$,$W(\lambda)$,$Q(\lambda)$ have the form $Q(x,\lambda) = \lambda - \sum_{i=1}^{n} q_i^2$, $a(\lambda) = \prod_{i=1}^{n}(\lambda - a_i)$ and

$$U(x,\lambda) = -a(\lambda)\left(1 + \frac{1}{2}\sum_{i=1}^{n}\frac{q_i^2}{(\lambda - a_i)}\right),$$

$$V(x,\lambda) = -\frac{1}{2}\frac{\mathrm{d}U(x,\lambda)}{\mathrm{d}x}, \quad W(x,\lambda) = a(\lambda)$$
$$\times\left(-\lambda + \frac{1}{2}\sum_{i=1}^{n} q_i^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\lambda - a_i}\left(p_i^2 + \frac{C_i^2}{q_i^2}\right)\right).$$

The dynamical system with Hamiltonian (11) is related to reduced Gaudin magnet *via* the identification

$$S_j^3 = p_j q_j,\ S_j^+ = q_j^2,\ S_j^- = -p_j^2 + \frac{C_j^2}{q_j^2},$$

and with $B(\lambda)$ given by

$$B(\lambda) = \begin{pmatrix} 0 & 1 \\ -\lambda + \frac{1}{2}\sum_{i=1}^{n} q_i^2 & 0 \end{pmatrix}. \tag{13}$$

The Lax representation yields the hyperelliptic curve $\det(L(\lambda) - \frac{1}{2}\nu\mathbf{1}_2) = 0$, where $\mathbf{1}_2$ is the 2×2 unit matrix. The moduli of the curve generate the integrals of motion $H, I_i, i = 1, \dots, n$,

$$\nu^2 = V^2(x,\lambda) + U(x,\lambda)W(x,\lambda). \tag{14}$$

The curve (14) can be written in canonical form as, $\nu^2 = 4\prod_{j=0}^{2n}(\lambda - \lambda_j)$, where $\lambda_j \neq \lambda_k$ are branching points.

From (14) and explicit expressions for $U(x,\lambda)$, $V(x,\lambda)$, $W(x,\lambda)$ we obtain

$$\nu^2 = a(\lambda)^2\left(\lambda - \sum_{i=1}^{n}\frac{I_i}{\lambda - a_i} - \frac{1}{4}\sum_{i=1}^{n}\frac{J_i^2}{(\lambda - a_i)^2}\right), \tag{15}$$

where $J_i = 2C_i$ and

$$I_i = \frac{1}{4}\sum_{k\neq i}\frac{1}{a_i - a_k}\left((q_i p_k - q_k p_i)^2 - \frac{C_i^2 q_k}{q_i^2} - \frac{C_k^2 q_i}{q_k^2}\right),$$
$$+ \frac{1}{2}p_i^2 - \frac{1}{2}a_i q_i^2 + \frac{1}{4}q_i^2\left(\sum_{k=1}^{n} q_k^2\right) + \frac{1}{2}\frac{C_i^2}{q_i^2}.$$

The parameters C_i are linked with the coordinates of the points $(a_i, \nu(a_i))$ by the formula

$$C_i^2 = -\frac{\nu(a_i)^2}{\prod_{k\neq i}(a_i - a_k)}, \tag{16}$$

where $i = 1, \dots n$. The solutions of the system (10) in terms of Novikov polynomials $F(x,\lambda)$ are given as

$$q_{\mathrm{i}}^2(x) = 2\frac{F(x, a_{\mathrm{i}} - \Delta)}{\prod_{k\neq \mathrm{i}}^{n}(a_{\mathrm{i}} - a_k)},\ \mathrm{i} = 1, \dots, n. \tag{17}$$

where we assume, without loss of generality, that the associated curve has the property $\tilde{c}_1 = 0$ and $\Delta = \frac{2}{2n+1}\sum_{i=1}^{n} a_i$. For generalized Garnier system the points a_i lie in the lacunae $[\lambda_{2i-1}, \lambda_{2i}], i = 1, \dots n$ and are branch points in the case of Garnier system (10) with $C_j^2 = 0, j = 1, \dots, n$.

3.2 $n+1$ dimensional generalized Hénon-Heiles type system

We consider a generalized Hénon-Heiles type system with $n+1$ degrees of freedom [2] with Hamiltonian

$$H = \frac{1}{2}\Big(\sum_{j=0}^{n} p_j^2\Big) + q_0^3 + \frac{1}{2}q_0\sum_{j=1}^{n} q_j^2$$
$$+ \frac{1}{4}\sum_{j=1}^{n}\left(a_j q_j^2 + \frac{C_j^2}{q_j^2}\right) - \frac{a_0}{4}q_0, \tag{18}$$

where $q_j, p_j, j = 0, \dots, n$ are the canonical coordinates and momenta and $a_0, C_j^2, a_j, j = 1, \dots, n$ are free constant parameters. The function H for $n = 1$ is the Hamiltonian of a classical integrable Hénon-Heiles system with the additional term C_1^2/q_1^2.

Next we will present (2×2) matrix Lax representation for generalized Hénon-Heiles system (18). The Lax representation have the form

$$L_x = [M(\lambda), L(\lambda)], \quad L = \begin{pmatrix} V & U \\ W & -V \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ Q & 0 \end{pmatrix} \tag{19}$$

where U, W, Q [9] are $Q(x,\lambda) = \lambda - q_1$, $a(\lambda) = \prod_{i=1}^{n}(\lambda - a_i)$ and

$$U(x,\lambda) = F(x,\lambda) = a(\lambda)\left(\lambda + \frac{1}{2}q_0 - \frac{1}{16}\sum_{j=1}^{n}\frac{q_j^2}{\lambda - a_j}\right),$$

$$V = -\frac{1}{2}F_x = a(\lambda)\left(-\frac{1}{4}p_0 + \frac{1}{16}\sum_{j}^{n}\frac{q_j p_j}{\lambda - a_j}\right),$$

$$W = -\frac{1}{2}F_{xx} + QF = a(\lambda)\left(\lambda^2 - \frac{1}{2}q_0\lambda\right) + a(\lambda)\left(\frac{1}{4}q_0^2 + \frac{1}{16}\sum_{j=1}^{n}q_j^2 - \frac{1}{16}a_0\right) + a(\lambda)\frac{1}{16}\left(\sum_{j=1}^{n}\frac{p_j^2}{\lambda - a_j} + \sum_{j=1}^{n}\frac{C_j^2}{q_j^2}\frac{1}{\lambda - a_j}\right).$$

The dynamical system with Hamiltonian (18) is related to reduced Gaudin magnet *via* the identification

$$S_j^3 = p_j q_j,\ S_j^+ = q_j^2,\ S_j^- = -p_j^2 + \frac{C_j^2}{q_j^2},$$

and with $B(\lambda)$ given by

$$B(\lambda) = \begin{pmatrix} -\frac{1}{4}p_0 & \lambda + \frac{1}{2}q_0 \\ W_1 & \frac{1}{4}p_0 \end{pmatrix}, \tag{20}$$

and $W_1 = \lambda^2 - \frac{1}{2}q_0\lambda + \frac{1}{4}q_0^2 + \sum_{j=1}^{n}\frac{1}{16}q_j^2 - \frac{1}{16}a_0$.

The corresponding algebraic curve of genus $n+1$ is

$$\nu^2 = a(\lambda)^2\left(\lambda^3 + \frac{1}{16}a_0\lambda + \frac{1}{8}H + \frac{1}{32}\sum_{i=1}^{n}\frac{H_i}{\lambda - a_i} + \frac{1}{256}\sum_{i=1}^{n}\frac{C_i^2}{(\lambda - a_i)^2}\right), \tag{21}$$

and

$$H_i = -p_0 q_i p_i + \frac{1}{8}a_0 q_i^2 + q_0\left(p_i^2 + \frac{C_i^2}{q_i^2}\right) - \frac{1}{4}q_0 a_i q_i^2 - \frac{1}{2}a_i\left(p_i^2 + \frac{C_i^2}{q_i^2}\right) - \frac{1}{2}a_i^2 q_i^2 - \frac{1}{8}\sum_{k\neq i}\frac{1}{a_i - a_k}\left((q_i p_k - q_k p_i)^2 - \frac{C_i^2 q_k}{q_i^2} - \frac{C_k^2 q_i}{q_k^2}\right).$$

The solutions of the system with Hamiltonian (18) in terms of Novikov polynomials $F(x,\lambda)$ are given as

$$q_0 = -u, \quad q_i^2(x) = 16\frac{F(x,a_i)}{\prod_{k\neq i}^{n}(a_i - a_k)},\ i = 1,\dots,n \tag{22}$$

where the points a_i lie in the lacunae $[\lambda_{2i-1}, \lambda_{2i}], i = 1,\dots n$ for generalized multidimensional Hénon-Heiles system and are branch points in the case multidimensional Hénon-Heiles system (18) with $C_j^2 = 0, j = 1,\dots,n$.

3.3 Neumann and Rosochatius system on the sphere

For the Rosochatius system the Hamiltonian is given by

$$H = \sum_{j=0}^{n}p_j^2 - \sum_{j=0}^{n}\left(a_j q_j^2 + \frac{\tilde{C}_j^2}{q_j^2}\right). \tag{23}$$

The Poisson bracket for this system is modified by constraining the particles to lie on the sphere, so that

$$(q,q) \equiv \sum_{j=0}^{n}q_j^2 = 1, \qquad (q,p) \equiv \sum_{j=0}^{n}q_j p_j = 0. \tag{24}$$

The Lax matrix for the Rosochatius system is defined by $Q(x,\lambda) = \lambda + 2\sum_{i=0}^{n}q_i^2$, $a(\lambda) = \prod_{i=0}^{n}(\lambda - a_i)$, and

$$U(x,\lambda) = a(\lambda)\left(\sum_{i=0}^{n}\frac{q_i^2}{\lambda - a_i}\right),\ V(x,\lambda) = -\frac{1}{2}U_x(x,\lambda)$$

$$W(x,\lambda) = a(\lambda)\left(1 - \sum_{i=0}^{n}\frac{1}{\lambda - a_i}(p_i^2 - \frac{\tilde{C}_i^2}{q_i^2})\right).$$

The dynamical system with Hamiltonian (23) is related to reduced Gaudin magnet *via* the identification

$$S_j^3 = p_j q_j, \quad S_j^+ = q_j^2, \quad S_j^- = -p_j^2 + \frac{\tilde{C}_j^2}{q_j^2},$$

and with $B(\lambda)$ given by

$$B(\lambda) = \begin{pmatrix} 0 & (q,q) \\ 0 & 0 \end{pmatrix} \tag{25}$$

where $\{\cdot\ ;\ \cdot\}$ in (2) is Dirac bracket. The solutions of the system with Hamiltonian (23) in terms of Novikov polynomials $F(x,\lambda)$ are given as [8]

$$q_i^2(x) = \frac{F(x,a_i)}{\prod_{k\neq i}^{n}(a_i - a_k)},\ i,k = 0,\dots,n \tag{26}$$

where the points $a_i, i = 0,\dots n$ lie in the lacunae $(-\infty, \lambda_0]$, $[\lambda_{2j-1}, \lambda_{2j}],\ j = 1,\dots n$ for Rosochatius system and are branch points in the case of Neumann system (23) with $C_j^2 = 0, j = 0,\dots,n$.

4 Conclusions

In this paper we have described a family of quasi-periodic, elliptic solutions for the dynamical systems related to KdV equation using a Lax pair method functions. General solutions are quasiperiodic due to quasiperiodic nature of Its-Matveev formula. Our approach is systematic in the sense that special solutions (periodic, "soliton", etc.) are obtained in a unified way.

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